Ideal magnetohydrodynamics and passive scalar motion as geodesics on semidirect product groups

1994 J. Phys. A: Math. Gen. 27 L21

(http://iopscience.iop.org/0305-4470/27/2/004)

J. Phys. A: Math. Gen. 27 (1994) L21–L25. Printed in the UK

LETTER TO THE EDITOR

Ideal magnetohydrodynamics and passive scalar motion as geodesics on semidirect product groups

Yuji Hattori

Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan

Received 18 November 1993

Abstract. Three-dimensional ideal magnetohydrodynamic (3d-iMHD) equations are shown to be a geodesic equation on an infinite-dimensional Lie group. The group is the semidirect product of a group of volume-preserving diffeomorphisms (particle motion) and a linear space of divergenceless vector fields (current density). The present theory includes the work of Zeitlin and Kambe for two-dimensional iMHD flow as a special case. Passive scalar motion in N-dimensional ideal hydrodynamic flow is also generally shown to be described by a geodesic equation.

It is well known that the motion of an inviscid incompressible flow is closely related to the Riemannian geometry of a group of volume-preserving diffeomorphisms [1]. That is, the motion of fluid particles is a geodesic curve with respect to the right-invariant metric; in other words, the geodesic equation is equivalent to the Euler equation for an ideal-hydrodynamics (iHD) flow. Mathematical formulation of the theory was given by Ebin and Marsden [2]. Arnold [1] calculated the curvature tensors and some sectional curvatures in the case of two-dimensional flow with periodic boundary and thereby tried to study the instability of the particle motion. The curvature tensors in the case of N-dimensional periodic flow were calculated by Lukatškii [3]. Applications to ABC flow [4] and stretching of line elements [5] have been exploited.

Recently Zeitlin and Kambe [6] have shown that the equations of two-dimensional ideal magnetohydrodynamics (2d-iMHD) are equivalent to the geodesic equation for the semidirect product of two infinite-dimensional groups. They dealt with the periodic flow and obtained curvature tensors in that case. They also considered '$2\frac{1}{2}$d-iHD' flow in relation to 2d-iMHD on the same semidirect product group but with a different metric.

In this letter, we show that the equations of 3d-iMHD are also described by a geodesic equation. This result is to give a new profound basis for MHD equations. Our theory includes the result of Zeitlin and Kambe [6] as a special case. The details and applications will be published in the forthcoming paper [7].

We consider a 3d-iMHD flow in a domain $M \in R^3$. For simplicity the flow domain M is assumed to be a flat torus with periodic boundary or a simply-connected finite region. In the former case both the velocity field u and the magnetic field B are assumed not to have a mean flux; i.e. $\int u \mathrm{d}^3x = \int B \mathrm{d}^3x = 0$. In the latter, the boundary ∂M of M is assumed to be perfectly conducting so that the magnetic field is tangent to the surface of M at the boundary ∂M and zero outside M. This implies the existence of surface current. Let us denote by SDiff(M) the group of volume-preserving diffeomorphisms on M. Its Lie algebra, which is denoted by $\mathrm{Vect}_0(M)$, is the tangent space of $\mathrm{SDiff}(M)$ at e (=identity mapping); it is a linear space of divergenceless vector fields on M,

$$\mathrm{Vect}_0(M) = \{v \in \mathrm{Vect}(M);\ \nabla \cdot v = 0\} \tag{1}$$

where the linear space of all vector fields is denoted by $\mathrm{Vect}(M)$.

Let us consider a semidirect product group of $\mathrm{SDiff}(M)$ and $\mathrm{Vect}_0(M)$ with the multiplication defined by

$$(g, \alpha) \circ (h, \beta) = (g \circ h, \mathrm{Ad}_{h^{-1}}\, \alpha + \beta) \tag{2}$$

where $g, h \in \mathrm{SDiff}(M)$ and $\alpha, \beta \in \mathrm{Vect}_0(M)$. The adjoint action Ad_h of $\mathrm{SDiff}(M)$ on $\mathrm{Vect}_0(M)$ is given by $\mathrm{Ad}_h\, \alpha = \tilde{L}_h \tilde{R}_{h^{-1}} \alpha$, where $\tilde{L}_h$ and $\tilde{R}_{h^{-1}}$ are the induced actions from the left and right actions of $\mathrm{SDiff}(M)$ on itself [8]. We denote by G this semidirect product group; $G = \mathrm{SDiff}(M) \rtimes \mathrm{Vect}_0(M)$. Its Lie algebra $g = T_{(e,0)}G$ (the tangent space of G at the identity $(e, 0)$) turns out to be

$$g = T_e\mathrm{SDiff}(M) \times \mathrm{Vect}_0(M) = \mathrm{Vect}_0(M) \times \mathrm{Vect}_0(M) \tag{3}$$

(note that $T_0\, \mathrm{Vect}_0(M) = \mathrm{Vect}_0(M)$ since $\mathrm{Vect}_0(M)$ is a linear space) with the bracket

$$[(u, \alpha), (v, \beta)] = ((u \cdot \nabla)v - (v \cdot \nabla)u, (u \cdot \nabla)\beta - (\beta \cdot \nabla)u + (\alpha \cdot \nabla)v - (v \cdot \nabla)\alpha) \tag{4}$$

for $u, v \in T_e\mathrm{SDiff}(M)$ and $\alpha, \beta \in \mathrm{Vect}_0(M)$. The Lie algebra g is later considered to be a space of velocity fields and current density fields.

Before discussing the Riemannian geometry of G, we should study the structure of the algebra of vector fields on G; we denote by $X(G)$ this algebra. We should remark the difference between $\mathrm{Vect}_0(M)$ and $X(G)$; a *vector* of $X \in X(G)$ at the identity $(e, 0)$ gives a *vector field* on M, an element of $\mathrm{Vect}_0(M)$; i.e. $X|_{(e,0)} \in \mathrm{Vect}_0(M)$. However, since the metric given below is right-invariant, it suffices to study the subalgebra $X^R(G)$ of $X(G)$, where $X^R(G)$ is a set of right-invariant vector fields on G. The right invariance of the metric reduces the task of the calculation that follows; we need to calculate the bracket, connection, etc, just at the identity. Moreover, since $X^R(G)$ is isomorphic to $g = T_{(e,0)}G$, its bracket immediately follows from that of g (equation (4)):

$$\begin{aligned}[(u^R, \alpha^R), (v^R, \beta^R)]|_{(h,\gamma)} &= \tilde{R}_{(h,\gamma)}\Big((u \cdot \nabla)v - (v \cdot \nabla)u \\ &\quad (u \cdot \nabla)\beta - (\beta \cdot \nabla)u + (\alpha \cdot \nabla)v - (v \cdot \nabla)\alpha \Big).\end{aligned} \tag{5}$$

Here $(u^R, \alpha^R) \in \mathrm{X}^R(G)$ is constructed from $(u, \alpha) \in g$ by the right action of G on g:

$$(u^R, \alpha^R)|_{(h,\gamma)} = \tilde{R}_{(h,\gamma)}(u, \alpha) = (u \circ h, \mathrm{Ad}_{h^{-1}}\, \alpha). \tag{6}$$

Now we introduce a right-invariant metric $\langle\ ,\ \rangle$ on G by

$$\langle (u, \alpha), (v, \beta) \rangle|_{(e,0)} = \int_M u \cdot v \mathrm{d}^3x + \int_M \alpha(-\Delta^{-1})\beta \mathrm{d}^3x \tag{7}$$

for $(u, \alpha), (v, \beta) \in T_{(e,0)}G$ and

$$\langle (u', \alpha'), (v', \beta') \rangle|_{(h,\gamma)} = \langle \tilde{R}_{(h^{-1}, -\mathrm{Ad}_h \gamma)}(u', \alpha'), \tilde{R}_{(h^{-1}, -\mathrm{Ad}_h \gamma)}(v', \beta') \rangle|_{(e,0)} \tag{8}$$

for $(u', \alpha'), (v', \beta') \in T_{(h,\gamma)}G$. Note that $(h,\gamma)^{-1} = (h^{-1}, -\mathrm{Ad}_h \gamma)$. This metric corresponds to the total energy of MHD flow since we regard u as a velocity field and α as a current density field. That is,

$$\langle (u,\alpha), (u,\alpha)\rangle|_{(e,0)} = \int_M u^2 \mathrm{d}^3x + \int_M B_\alpha^2 \mathrm{d}^3x \tag{9}$$

where $\nabla \times B_\alpha = \alpha$, $\times$ denoting the cross product in R^3. Note that we should include surface current in equation (7) to derive equation (9).

For a given metric there is a Levi-Civita connection $\tilde{\nabla}$ derived by the following formula [9]

$$\begin{aligned} 2\langle \tilde{\nabla}_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle - \langle X, [Y, Z]\rangle \end{aligned} \tag{10}$$

where $X, Y, Z \in \mathrm{X}(G)$. Thus we obtain the following expression of the Levi-Civita connection for right-invariant fields using equations (5), (7), (8) and (10):

$$\begin{aligned} \tilde{\nabla}_{(u^R,\alpha^R)}(v^R, \beta^R)|_{(h,\gamma)} = \tilde{R}_{(h,\gamma)} \Big(& P[(u\cdot\nabla)v - \tfrac{1}{2}(\alpha \times B_\beta + \beta \times B_\alpha)], \\ & P\Big[\tfrac{1}{2}\nabla \times (-u \times \beta + v \times \alpha) - \tfrac{1}{2}\nabla \times (\nabla \times (u \times B_\beta + v \times B_\alpha))\Big] \Big). \end{aligned} \tag{11}$$

Here the projection operator from $\mathrm{Vect}(M)$ to $\mathrm{Vect}_0(M)$ is denoted by P; every vector field w on M is uniquely decomposed as

$$w = u + \nabla f$$

where $u \in \mathrm{Vect}_0(M)$ and f is a function on M satisfying $\int_M f \mathrm{d}^3x = 0$; the projection is now represented as $P[w] = u$.

A curve $\sigma(t)$ on G is called a geodesic when it satisfies

$$\tilde{\nabla}_X X = 0 \tag{12}$$

where $X = \frac{d}{dt}\sigma(t) \in T_{\sigma(t)}G$. Since the vector field X on curve σ is not right-invariant in general, we write equation (12) in the form

$$\frac{\partial X}{\partial t} + \tilde{\nabla}_{X'_t} X'_t = 0. \tag{13}$$

Here we have introduced the right-invariant vector field X'_t defined by

$$X'_{t_0}(\sigma(t_0)) = X(t_0) \qquad X'_{t_0}|_{(g,\alpha)} = \tilde{R}_{(g\cdot\sigma(t_0)^{-1},\alpha)} X(t_0) \tag{14}$$

for fixed t_0. Equation (13) is easily checked using equation (14) [7]. Using the Levi-Civita connection (11) and applying $\tilde{R}_{\sigma(t)^{-1}}$ to equation (13) in order to express the equations in terms of $T_{(e,0)}G$, we obtain

$$\frac{\partial u}{\partial t} + P[(u\cdot\nabla)u - \alpha \times B_\alpha] = 0 \tag{15a}$$

$$\frac{\partial \alpha}{\partial t} - P[\nabla \times (\nabla \times (u \times B_\alpha))] = 0 \tag{15b}$$

for $(u,\alpha)=\tilde{R}_{\sigma(t)^{-1}}X(t)\in T_{(e,0)}G$. If we write the projection explicitly as $P[w]=w-\nabla p$ with a function p on M and integrate equation (15*b*) into that for B_α, the above equations turn out to be the 3d-iMHD equations

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u=-\nabla p+j\times B \tag{16a}$$

$$\frac{\partial B}{\partial t}=\nabla\times(u\times B) \tag{16b}$$

where $j=\alpha, B=B_\alpha$. The conditions $\nabla\cdot u=\nabla\cdot B=0$ are already assumed effective as we are working with $T_{(e,0)}G=\mathrm{Vect}_0(M)\times\mathrm{Vect}_0(M)$. Therefore, the three-dimensional ideal MHD motion is proved to be a geodesic motion on the semidirect product group $\mathrm{SDiff}(M)\rtimes\mathrm{Vect}_0(M)$.

We can derive explicit expressions for curvature tensors and some sectional curvatures for a specific case such as $M=T^3$ (three-dimensional periodic flow; T^N denotes the N-dimensional flat torus). They will be described in the forthcoming paper [7].

The above result is a generalization of that of Zeitlin and Kambe [6]. Let us see this in the following. We consider the periodic flow ($M=T^3$) and define the subgroup G_{2d} of G by

$$G_{2d}=\{(g,\alpha)\in G;\,g=(g_x(x,y),g_y(x,y),z),\alpha=(0,0,\alpha(x,y))\}. \tag{17}$$

The multiplication in this subgroup follows from that of the entire group G of (2):

$$(g,\alpha)\circ(h,\beta)=(g\circ h,(0,0,\alpha(h(x,y))+\beta(x,y)). \tag{18}$$

Note that for a vector field α directed in z axis the adjoint action reduces to $Ad_{h^{-1}}\alpha=\tilde{L}_{h^{-1}}\tilde{R}_h\alpha=\tilde{R}_h\alpha$, since the diffeomorphism h is a pure two-dimensional one. Thus the subgroup G_{2d} can be identified with the semidirect product group of $\mathrm{SDiff}(M')$ and $F(M')$ (= the linear space of functions on $M'=T^2$) with the following multiplication

$$(g,\alpha)\circ(h,\beta)=(g\circ h,\alpha\circ h+\beta). \tag{19}$$

Zeitlin and Kambe [6] worked on this group and obtained 2d-iMHD equations with the metric (7),(8) and the Levi-Civita connection. They interpreted multiplication (19) as charge transport. In the present theory, multiplication (2) immediately implies adjoint transport of current, though simpler interpretation may be found.

Zeitlin and Kambe [6] also considered '$2\frac{1}{2}$d-iHD motion' in T^2 with the same semidirect product group but with a different metric. '$2\frac{1}{2}$d-iHD motion' is actually a passive scalar motion in 2d-iHD flow. We can generalize their result for $2\frac{1}{2}$d-iHD motion to passive scalar motion in N-dimensional iHD flow in any domain in the same fashion as 3d-iMHD case. Let M be a flow domain in R^N. As before, we assume M to be a flat torus with periodic boundary or a simply-connected finite region. We introduce the semidirect product group $H=\mathrm{SDiff}(M)\rtimes F(M)$ with multiplication (19). The bracket of right-invariant vector fields on H becomes

$$[(u^R,\alpha^R),(v^R,\beta^R)]|_{(h,\gamma)}=\tilde{R}_{(h,\gamma)}((u\cdot\nabla)v-(v\cdot\nabla)u,(u\cdot\nabla)\beta-(v\cdot\nabla)\alpha) \tag{20}$$

where $(u^R,\alpha^R)|_{(h,\gamma)}=\tilde{R}_{(h,\gamma)}(u,\alpha)=(u\circ h,\alpha\circ h)$ for $(u,\alpha)\in T_{(e,0)}H$. If we define the right-invariant metric by

$$\langle(u,\alpha),(v,\beta)\rangle|_{(e,0)}=\int_M u\cdot v\,\mathrm{d}^Nx+\int_M\alpha\beta\,\mathrm{d}^Nx \tag{21}$$

with equation (8), the Levi-Civita connection is derived by formula (10) as

$$\tilde{\nabla}_{(u^R,\alpha^R)}(v^R,\beta^R)|_{(h,\gamma)} = \tilde{R}_{(h,\gamma)}(P[(u\cdot\nabla)v],\ (u\cdot\nabla)\beta)). \tag{22}$$

Hence the geodesic equation becomes

$$\frac{\partial u}{\partial t} + P[(u\cdot\nabla)u] = 0 \tag{23a}$$

$$\frac{\partial \alpha}{\partial t} + (u\cdot\nabla)\alpha = 0. \tag{23b}$$

The above equations are just the Euler equation (23*a*) for incompressible inviscid fluid and the kinematic equation (23*b*) for passive scalar field α.

Finally we refer to the relation between our semidirect product structure and that in Marsden *et al* [10]. Semidirect products appear quite frequently in ideal continuum mechanics. They are summarized in terms of Hamiltonian mechanics on Lie groups in Marsden *et al* [10]. They treated the heavy top, compressible fluids, magnetohydrodynamics, elasticity, the Maxwell–Vlasov equations and multifluid plasmas.

There is an apparent difference in the choice of essential variables. Marsden *et al* worked on the following algebra in the case of compressible IMHD flow (using the present notation [10]):

$$\mathrm{Vect}(M)\times \mathrm{F}(M)\times \Lambda^1(M). \tag{24}$$

Here the set of one-form fields on M is denoted by $\Lambda^1(M)$. The dual space of $F(M)$ and that of $\Lambda^1(M)$ were identified with the space of mass density fields and that of magnetic fields, respectively. Thus they considered the *magnetic field* as one of the essential variables. However, we have so far regarded $T_{(e,0)}G$ as the product of the space of velocity fields and that of *current density fields*. Further discussions on this point will be developed elsewhere.

The author expresses his gratitude to Professors T Kambe and V J Zeitlin for helpful guidance.

References

[1] Arnold V I 1966 *Ann. Inst. Fourier, Grenoble* **16** 319; 1978 *Mathematical Methods of Classical Mechanics* appendix 2 (New York: Springer)
[2] Ebin D and Marsden J 1970 *Ann. Math.* **90** 102
[3] Lukatskii A M 1981 *Russ. Math. Surv.* **36** 179
[4] Nakamura F, Hattori Y and Kambe T 1992 *J. Phys. A: Math. Gen.* **25** L45
[5] Hattori Y and Kambe T 1993 *Fluid Dyn. Res.* to appear
[6] Zeitlin V and Kambe T 1993 *J. Phys. A: Math. Gen.* **26** 5025
[7] Hattori Y in preparation
[8] Abraham R and Marsden J E 1978 *Foundations of Mechanics* 2nd ed (Reading, MA: Addison-Wesley)
[9] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* (New York: Wiley)
[10] Marsden J E, Ratiu T and Weinstein A 1984 *Trans. Am. Math. Soc.* **281** 147